A Proof: Proof questions are where you are given a result and asked to demonstrate that the result is true.

Example 1

A cylinder has a surface area of 160 cm^2. Show that the maximum volume will occur when the height is twice the length of the radius.

Surface area of the cylinder: $A = 2\pi r^2 + 2\pi r h$

(r = radius and h = height)

If the surface area is 160 cm^2, then: $160 = 2\pi r^2 + 2\pi r h$

Re-arranging to make h the subject: $h = \frac{80}{\pi r} - r$

Volume of the cylinder is given by: $V = \pi r^2 h$

In terms of radius only: $V = \pi r^2 \left(\frac{80}{\pi r} - r\right)$

$$V = 80r - \pi r^3$$

To find the maximum volume, differentiate and let the derivative equal zero:

$$\frac{dV}{dr} = 80 - 3\pi r^2$$

$$\Rightarrow \frac{dV}{dr} = 0$$

$$\Rightarrow r = \sqrt{\frac{80}{3\pi}}$$

Substituting this result into the expression for h:

$$h = \frac{80}{\pi} \div \sqrt{\frac{80}{3\pi}} - \sqrt{\frac{80}{3\pi}}$$

$$h = \frac{3\sqrt{80}\sqrt{3\pi} - \sqrt{80}\sqrt{3\pi}}{3\pi}$$

$$h = \frac{2\sqrt{80}\sqrt{3\pi}}{3\pi}$$

$$h = 2\sqrt{\frac{80}{3\pi}}$$

$$h = 2r$$

Therefore the maximum volume occurs when the height is twice the radius.

Example 2

If the function $y = ax^2 + bx + c$ has two distinct roots, show that the turning point has the x-value at the midpoint between these two roots.

$y = ax^2 + bx + c$

$y' = 2ax + b$

For turning point $y' = 0$, therefore:

$$2ax + b = 0$$

$$\Rightarrow x = \frac{-b}{2a}$$

The roots of the function are:

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The midpoint between these two roots will be their average.

Thus the average of the two roots will be:

$$\frac{1}{2}\left[\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right]$$

$$= \frac{1}{2}\left[\frac{-2b}{2a}\right]$$

$$= \frac{-b}{2a}$$

This is the same value as the x-value of the turning point established above, hence the turning point occurs at the midpoint between the two roots.